MICHIGAN STATE UNIVERSITY

DEPARTMENT OF STATISTICS

ON A PARTIAL DIFFERENTIAL EQUATION OF EPIDEMIC THEORY. II -

THE MODEL WITH IMMIGRATION

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1. Introduction

A generalization of the basic model which arises in the stochastic theory of epidemics, discussed by Gani [1], is that for which at time $t>0$, there are in circulation r uninfected susceptibles and s infectives, $n + m + a - r - s$ removals of infectives having occurred since time $t = 0$. Here the integer m represents the number of susceptible immigrants who have entered the population since time $t = 0$, when the initial population consisted of n susceptibles and a infectives only.

It will be found useful, instead of considering as before the probabilities

$$p_{rs}(t) = \Pr\left\{ r \text{ susceptibles and } s \text{ infectives at } t>0 \,\middle|\, n \text{ susceptibles and } a \text{ infectives at } t = 0 \right\},$$

to keep track of the number of immigrants m in the population as well. Thus, we now consider the probabilities

$$p_{rsm}(t) = \Pr\left\{ r \text{ susceptibles, } s \text{ infectives and } m \text{ immigrants at } t>0 \,\middle|\, n \text{ susceptibles, } a \text{ infectives and } 0 \text{ immigrants at } t = 0 \right\},$$

and denote by β the infection parameter, by γ the removal parameter and by μ the immigration parameter, precisely as in Bailey [2](p.179).

Then if $\rho = \gamma/\beta$, and $\sigma = \mu/\beta$, we find after the usual change in time scale that the probabilities $p_{rsm}(t)$ satisfy the differential-difference equations

$$\text{(1.1)} \qquad \frac{dp_{rsm}}{dt} = (r+1)(s-1)p_{r+1,s-1,m} - \{s(r+\rho)+\sigma\}p_{rsm} + \rho(s+1)p_{r,s+1,m} + \sigma p_{r-1,s,m-1}$$

where $0 \leq s \leq m+n+a-r$, $0 \leq r \leq m+n$, $0 \leq m < \infty$, with the initial condition $p_{na0}(0) = 1$.

It is readily seen that the probability generating function (p.g.f.) of the process

$$\text{(1.2)} \qquad \Pi(z,w,v,t) = \sum_{r,s,m} p_{rsm}(t)z^r w^s v^m \qquad (|z|,|w|,|v| \leq 1)$$

satisfies the partial differential equation

$$\text{(1.3)} \qquad \frac{\partial \Pi}{\partial t} = w(w-z)\frac{\partial^2 \Pi}{\partial z \partial w} + \rho(1-w)\frac{\partial \Pi}{\partial w} + \sigma(zv-1)\Pi.$$

We note that with $v = 1$, when the immigrant number is not taken into account, this equation is identical with the more usual one given by Bartlett [3] or Bailey [2]. With $\sigma = 0$, (1.3) reduces to the equation previously solved in [1].

Using a method related to that outlined in [1], we shall in this paper solve equation (1.3) formally in the general case.

2. Remarks on the processes with and without immigration.

Let us, much as in [1], write the p.g.f. (1.2) in the form

$$\Pi(z,w,v,t) = \sum_{m=0}^{\infty} v^m \sum_{r=0}^{m+n} \sum_{s=0}^{m+n+a-r} p_{rsm}(t) z^r w^s \tag{2.1}$$
$$= \sum_{m=0}^{\infty} v^m g_m(z,w,t) .$$

On substituting this expression in (1.3) we obtain

$$\sum_{m=0}^{\infty} v^m \frac{\partial g_m}{\partial t} = \tag{2.2}$$
$$w(w-z) \sum_{m=0}^{\infty} v^m \frac{\partial^2 g_m}{\partial z \partial w} + \rho(1-w) \sum_{m=0}^{\infty} v^m \frac{\partial g_m}{\partial w} + \sigma z \sum_{m=0}^{\infty} v^{m+1} g_m - \sigma \sum_{m=0}^{\infty} v^m g_m .$$

Since this is true for all v, by equating coefficients of v^m on both sides of the equation we obtain

$$\frac{\partial g_0}{\partial t} = w(w-z)\frac{\partial^2 g_0}{\partial z \partial w} + \rho(1-w)\frac{\partial g_0}{\partial w} - \sigma g_0 \tag{2.3}$$
$$\frac{\partial g_m}{\partial t} = w(w-z)\frac{\partial^2 g_m}{\partial z \partial w} + \rho(1-w)\frac{\partial g_m}{\partial w} - \sigma g_m + \sigma z g_{m-1} \quad (m=1,2,\ldots).$$

subject to the initial conditions $g_0(z,w,0) = z^n w^a$, $g_m(z,w,0) = 0$ for $m > 0$.

We can simplify these slightly by setting

$$h_m(z,w,t) = e^{\sigma t} g_m(z,w,t) \tag{2.4}$$

[illegible] values of $m \geq 0$. Then (2. [illegible]) can be rewritten as

(2.5) [illegible]

[illegible]

[illegible] that satisfy [1] [illegible] without integration. Thus $h_0(z,w,t)$ is [illegible] and it [illegible] in a form enabling us to [illegible] for the [illegible] solution of the $h_n(z,w,t)$ $(n = 1,2,\ldots)$ which we [illegible]

Let us write $h_0(z,w,t)$ as

(2.6) [illegible]

Then we have seen in [1] that

(2.7)
$$\frac{\partial f_{0r}}{\partial t} = [illegible] \frac{\partial f_{0r}}{\partial w} [illegible]$$
$$\frac{\partial f_{0n}}{\partial t} = -((n+\rho)w - \rho) \frac{\partial f_{0n}}{\partial w}$$

subject to the initial conditions $f_{0n}(w,0) = w^n$, $f_{0r}(w,0) = $ [illegible]

Taking Laplace transforms

$$F_{0r}(w,s) = \int_0^\infty e^{-st} f_{0r}(w,t)\,dt$$

with respect to time, we find that the equations (2.7) reduce to

$$sF_{0r}(w,s) = w^2(r+1)\frac{\partial F_{0,r+1}}{\partial w} - ((r+\rho)w-\rho)\frac{\partial F_{0r}}{\partial w} \quad (r=0,1,\dots,n-1)$$

(2.8)

$$sF_{0n}(w,s) - w^n = -((n+\rho)w-\rho)\frac{\partial F_{0n}}{\partial w} .$$

The solutions to these are found to be

$$F_{0n}(w,s) = \sum_{j=0}^{n}\binom{n}{j}\left(\frac{\rho}{n+\rho}\right)^{n-j}\left(w-\frac{\rho}{n+\rho}\right)^{j}(s+(n+\rho)j)^{-1}$$

(2.9)

$$= \sum_{j=0}^{n}\left(w-\frac{\rho}{n+\rho}\right)^{j}F_{0n}^{(j)}\left(\frac{\rho}{n+\rho},s\right)/j!$$

and consecutively

(2.10) $$F_{0r}(w,s) = \sum_{j=0}^{n+n-r}\left(w-\frac{\rho}{r+\rho}\right)^{j}F_{0r}^{(j)}\left(\frac{\rho}{r+\rho},s\right)/j!$$

where the coefficients $F_{0r}^{(j)}(\frac{\rho}{r+\rho},s)$ can be expressed in terms of the previously known $F_{0,r+1}(w,s)$ as

$$F_{0r}^{(j)}\left(\frac{\rho}{r+\rho},s\right) = (r+1)(s+(r+\rho)j)^{-1}\times$$

(2.11)

$$\left\{j(j-1)F_{0,r+1}^{(j-1)}\left(\frac{\rho}{r+\rho},s\right) - 2j\left(\frac{\rho}{r+\rho}\right)F_{0,r+1}^{(j)}\left(\frac{\rho}{r+\rho},s\right) + \left(\frac{\rho}{r+\rho}\right)^{2}F_{0,r+1}^{(j+1)}\left(\frac{\rho}{r+\rho},s\right)\right\}.$$

We now proceed to use these known solutions for $h_0(w,t)$ as the bases of the explicit solutions for the $h_m(z,w,t)$ $(m = 1,2,\dots)$.

3. Method of solution of the partial differential equations

Let us first consider the equation

$$(3.1)\qquad \frac{\partial h_1}{\partial t} = w(w-z)\frac{\partial^2 h_1}{\partial z\,\partial w} + \rho(1-w)\frac{\partial h_1}{\partial w} + \sigma z h_0 ,$$

and write $h_1(z,w,t)$ as

$$(3.2)\qquad h_1(z,w,t) = \sum_{r=0}^{n+1} z^r f_{1r}(w,t) .$$

Substituting (3.2) in (3.1), and recalling that $h_0(z,w,t) = \sum_{r=0}^{n} z^r f_{0r}(w,t)$, we obtain

$$(3.3)\qquad \begin{aligned} \frac{\partial f_{1,n+1}}{\partial t} &= -((n+1+\rho)w-\rho)\frac{\partial f_{1,n+1}}{\partial w} + \sigma f_{0n} \\ \frac{\partial f_{1r}}{\partial t} &= w^2(r+1)\frac{\partial f_{1,r+1}}{\partial w} - ((r+\rho)w-\rho)\frac{\partial f_{1r}}{\partial w} + \sigma f_{0,r-1} \quad (r=0,1,\ldots,n) \end{aligned}$$

where $f_{0,-1}(w,t)$ is understood to be zero.

If, as before, we take Laplace transforms with respect to time, these equations reduce to

$$(3.4)\qquad \begin{aligned} sF_{1,n+1}(w,s) &= -((n+1+\rho)w-\rho)\frac{\partial F_{1,n+1}}{\partial w} + \sigma F_{0n} \\ sF_{1r}(w,s) &= w^2(r+1)\frac{\partial F_{1,r+1}}{\partial w} - ((r+\rho)w-\rho)\frac{\partial F_{1r}}{\partial w} + \sigma F_{0,r-1} \quad (r=0,1,\ldots,n). \end{aligned}$$

We note that, except for the last term, the equations are similar in form to those for the F_{0r}; we would therefore expect similar methods to apply in their solution.

Thus, for $F_{1,n+1}$, assuming for simplicity that s is real, we may rewrite the equation (3.4) as

$$(3.5)\quad \frac{\partial}{\partial w}\left\{((n+1+\rho)w-\rho)^{\frac{s}{n+1+\rho}} F_{1,n+1}\right\} = ((n+1+\rho)w-\rho)^{\frac{s}{n+1+\rho}-1}\sigma F_{0n}\ ,$$

or

$$(3.6)\quad (n+1+\rho)\frac{\partial}{\partial w}\left\{\left(w-\frac{\rho}{n+1+\rho}\right)^{\frac{s}{n+1+\rho}} F_{1,n+1}\right\} = \sigma\sum_{j=0}^{a}\left(w-\frac{\rho}{n+1+\rho}\right)^{j-1+\frac{s}{n+1+\rho}} F_{0n}^{(j)}\left(\frac{\rho}{n+1+\rho},s\right)/j!$$

On integration, with due regard for the initial condition $\lim_{s\to\infty} sF_{1,n+1}(w,s) = 0$ we obtain

$$(3.7)\quad \left(w-\frac{\rho}{n+1+\rho}\right)^{\frac{s}{n+1+\rho}} F_{1,n+1} = \sigma\sum_{j=0}^{a}\left(w-\frac{\rho}{n+1+\rho}\right)^{j+\frac{s}{n+1+\rho}}(s+j(n+1+\rho))^{-1}F_{0n}^{(j)}\left(\frac{\rho}{n+1+\rho},s\right)/j!$$

Hence

$$(3.8)\quad \begin{aligned} F_{1,n+1}(w,s) &= \sigma\sum_{j=0}^{a}\left(w-\frac{\rho}{n+1+\rho}\right)^{j}(s+j(n+1+\rho))^{-1}F_{0n}^{(j)}\left(\frac{\rho}{n+1+\rho},s\right)/j! \\ &= \sum_{j=0}^{a}\left(w-\frac{\rho}{n+1+\rho}\right)^{j} F_{1,n+1}^{(j)}\left(\frac{\rho}{n+1+\rho},s\right)/j! \end{aligned}$$

where the

$$F_{1,n+1}^{(j)}\left(\frac{\rho}{n+1+\rho},s\right) = \sigma(s+j(n+1+\rho))^{-1}F_{0n}^{(j)}\left(\frac{\rho}{n+1+\rho},s\right)$$

and the $F_{0n}^{(k)}\left(\frac{\rho}{n+1+\rho},s\right)$ can be readily obtained from the known result for $F_{0n}(w,s)$; in particular, the derivatives are

$$(3.9)\quad F_{0n}^{(k)}\left(\frac{\rho}{n+1+\rho},s\right) = \sum_{j=k}^{a}\frac{a!}{(a-j)!(j-k)!}\left(\frac{\rho}{n+\rho}\right)^{a-j}\left(\frac{-\rho}{(n+\rho)(n+1+\rho)}\right)^{j-k}(s+j(n+\rho))^{-1}.$$

We now proceed to obtain all remaining $F_{1r}(w,s)$, $r = n, n-1, \ldots, 0$. Let us assume for the moment that for any of these particular values of

r, $F_{1,r+1}(w,s)$ is known, and can be expressed in the form

$$F_{1,r+1}(w,s) = \sum_{j=0}^{a+n-r} \left(w - \frac{\rho}{r+1+\rho}\right)^j F_{1,r+1}^{(j)}\left(\frac{\rho}{r+1+\rho}, s\right)/j!$$

(3.10)

$$= \sum_{j=0}^{a+n-r} \left(w - \frac{\rho}{r+\rho}\right)^j F_{1,r+1}^{(j)}\left(\frac{\rho}{r+\rho}, s\right)/j!$$

where

$$F_{1,r+1}^{(k)}\left(\frac{\rho}{r+\rho}, s\right) =$$

(3.11)

$$\sum_{j=k}^{a+n-r} \left(\frac{\rho}{(r+\rho)(r+1+\rho)}\right)^{j-k} F_{1,r+1}^{(j)}\left(\frac{\rho}{r+1+\rho}, s\right)/(j-k)!$$

Then, the partial differential equation (3.4) for $F_{1r}(w,s)$ can be written as

$$(r+\rho)\frac{\partial}{\partial w}\left\{\left(w - \frac{\rho}{r+\rho}\right)^{\frac{s}{r+\rho}} F_{1r}\right\} = \left(w - \frac{\rho}{r+\rho}\right)^{\frac{s}{r+\rho}-1} \times$$

(3.12)

$$\left\{(r+1)\sum_{i=0}^{2}\sum_{j=1}^{a+n-r}\left(w - \frac{\rho}{r+\rho}\right)^{j+1-i}\binom{2}{i}\left(\frac{\rho}{r+\rho}\right)^i F_{1,r+1}^{(j)}\left(\frac{\rho}{r+\rho}, s\right)/(j-1)! + \sigma\sum_{j=0}^{a+n-r+1}\left(w - \frac{\rho}{r+\rho}\right)^j F_{0,r-1}^{(j)}\left(\frac{\rho}{r+\rho}, s\right)/j!\right\}$$

in which both w^2 and $F_{0,r-1}(w,s)$ have been expressed as polynomials in $(w - \frac{\rho}{r+\rho})$. On integration, with due regard for the initial condition $\lim_{s\to\infty} sF_{1r}(w,s) = 0$, we obtain

$$F_{1r}(w,s) = \left\{(r+1)\sum_{i=0}^{2}\sum_{j=1}^{a+n-r}\left(w - \frac{\rho}{r+\rho}\right)^{j+1-i}\left(s+(r+\rho)(j-i+1)\right)^{-1}\binom{2}{i}\left(\frac{\rho}{r+\rho}\right)^i F_{1,r+1}^{(j)}\left(\frac{\rho}{r+\rho}, s\right)/(j-1)!\right.$$

(3.13)

$$\left. + \sigma\sum_{j=0}^{a+n-r+1}\left(w - \frac{\rho}{r+\rho}\right)^j\left(s+(r+\rho)j\right)^{-1} F_{0,r-1}^{(j)}\left(\frac{\rho}{r+\rho}, s\right)/j!\right\}$$

$$= \sum_{k=0}^{a+n-r+1}\frac{1}{k!}\left(w - \frac{\rho}{r+\rho}\right)^k\left(s+(r+\rho)k\right)^{-1}\left\{(r+1)\left[k(k-1)F_{1,r+1}^{(k-1)}\left(\frac{\rho}{r+\rho}, s\right) + \right.\right.$$

$$\left.\left. 2k\left(\frac{\rho}{r+\rho}\right)F_{1,r+1}^{(k)}\left(\frac{\rho}{r+\rho}, s\right) + \left(\frac{\rho}{r+\rho}\right)^2 F_{1,r+1}^{(k+1)}\left(\frac{\rho}{r+\rho}, s\right)\right] + \sigma F_{0,r-1}^{(k)}\left(\frac{\rho}{r+\rho}, s\right)\right\}$$

$$= \sum_{k=0}^{a+n-r+1}\left(w - \frac{\rho}{r+\rho}\right)^k F_{1r}^{(k)}\left(\frac{\rho}{r+\rho}, s\right)/k!$$

where

$$F_{1r}^{(k)}\left(\frac{\rho}{r+\rho}, s\right) = (s+(r+\rho)k)^{-1}\Big\{(r+1)\Big[k(k-1)F_{1,r+1}^{(k-1)}\left(\frac{\rho}{r+\rho}, s\right) +$$

$$(3.14)\quad 2k\left(\frac{\rho}{r+\rho}\right)F_{1,r+1}^{(k)}\left(\frac{\rho}{r+\rho}, s\right) + \left(\frac{\rho}{r+\rho}\right)^2 F_{1,r+1}^{(k+1)}\left(\frac{\rho}{r+\rho}, s\right)\Big] + \sigma F_{0,r-1}^{(k)}\left(\frac{\rho}{r+\rho}, s\right)\Big\}.$$

[illegible] the $F^{(k)}_{1,r+1}(\frac{\rho}{r+\rho}, s)$ are known from (3.11), and the $F^{(k)}_{0,r-1}(\frac{\rho}{r+\rho}, s)$ from the general expressions for $F_{0r}(w,s)$ as in (2.10). We note that F_{1r} [illegible] is also of the form assumed for $F_{1,r+1}(w,s)$ in (3.10) and since $F_{1,n+1}(w,s)$ has been shown to be of this form, then it follows by induction that (3.13) will be the form of all the $F_{1r}(w,s)$ ($r = n+1,$ [illegible]); the equation (3.14) shows the relation between coefficients of consecutive polynomials in w. It should be pointed out that the form (3.13) includes (3.8) since $F^{(k)}_{1r}(w,s) = 0$ for $r > n+1$.

4. The general solution

We may now in the same way proceed to solve the general equation

$$(4.1)\quad \frac{\partial h_m}{\partial t} = w(w-z)\frac{\partial^2 h_m}{\partial z \partial w} + \rho(1-w)\frac{\partial h_m}{\partial w} + \sigma z h_{m-1}$$

for h_m $(m > 1)$, assuming that $h_{m-1} = \sum_{r=0}^{n+m-1} z^r f_{m-1,r}(w,t)$ is known and starting with $h_2(z,w,t)$. We first write

$$(4.2)\quad h_m(z,w,t) = \sum_{r=0}^{n+m} z^r f_{mr}(w,t)$$

which from (4.1) leads to

$$(4.3)\quad \begin{aligned} \frac{\partial f_{m,n+m}}{\partial t} &= -((n+m+\rho)w-\rho)\frac{\partial f_{m,n+m}}{\partial w} + \sigma f_{m-1,n+m-1} \\ \frac{\partial f_{mr}}{\partial t} &= w^2(r+1)\frac{\partial f_{m,r+1}}{\partial w} - ((r+\rho)w-\rho)\frac{\partial f_{mr}}{\partial w} + \sigma f_{m-1,r-1} \quad (r=0,1,\dots,n+m-1) \end{aligned}$$

where $f_{m-1,-1}(w,t)$ is again understood to be zero.

These equations reduce, after taking Laplace transforms with respect to time to

$$
(4.4)\qquad
\begin{aligned}
sF_{m,n+m} &= -((n+m+\rho)w-\rho)\frac{\partial F_{m,n+m}}{\partial w} + \sigma F_{m-1,n+m-1} \\
sF_{mr} &= w^2(r+1)\frac{\partial F_{m,r+1}}{\partial w} - ((r+\rho)w-\rho)\frac{\partial F_{mr}}{\partial w} + \sigma F_{m-1,r-1} \quad (r=0,1,\dots,n+m-1),
\end{aligned}
$$

and the methods of Section 3 lead directly to a solution

$$
(4.5)\qquad
\begin{aligned}
F_{mr}(w,s) = \sum_{k=0}^{n+a+m-r}\frac{1}{k!}\left(w-\frac{\rho}{r+\rho}\right)^k (s+(r+\rho)k)^{-1}\Big\{(r+1)\Big[k(k-1)F_{m,r+1}^{(k-1)}\left(\frac{\rho}{r+\rho},s\right)+ \\
2k\left(\frac{\rho}{r+\rho}\right)F_{m,r+1}^{(k)}\left(\frac{\rho}{r+\rho},s\right)+\left(\frac{\rho}{r+\rho}\right)^2 F_{m,r+1}^{(k+1)}\left(\frac{\rho}{r+\rho},s\right)\Big]+\sigma F_{m-1,r-1}^{(k)}\left(\frac{\rho}{r+\rho},s\right)\Big\}
\end{aligned}
$$

where for $r = m + n$ only the last term within the curly brackets, $\left\{\sigma F_{m-1,m+n-1}^{(k)}\left(\frac{\rho}{r+\rho},s\right)\right\}$, is non-zero, and for $r = 0$ all $F_{m-1,-1}^{(k)} = 0$.

We note that

$$
(4.6)\qquad
\begin{aligned}
F_{mr}^{(k)}\left(\frac{\rho}{r+\rho},s\right) = (s+(r+\rho)k)^{-1}\Big\{(r+1)\Big[k(k-1)F_{m,r+1}^{(k-1)}\left(\frac{\rho}{r+\rho},s\right)+ \\
2k\left(\frac{\rho}{r+\rho}\right)F_{m,r+1}^{(k)}\left(\frac{\rho}{r+\rho},s\right)+\left(\frac{\rho}{r+\rho}\right)^2 F_{m,r+1}^{(k+1)}\left(\frac{\rho}{r+\rho},s\right)\Big]+\sigma F_{m-1,r-1}^{(k)}\left(\frac{\rho}{r+\rho},s\right)\Big\},
\end{aligned}
$$

where, if $F_{m-1,r-1}(w,s)$ and $F_{m,r+1}(w,s)$ are known and expressed in their usual forms as

$$
(4.7)\qquad
\begin{aligned}
F_{m-1,r-1}(w,s) &= \sum_{k=0}^{n+a+m-r}\left(w-\frac{\rho}{r-1+\rho}\right)^k F_{m-1,r-1}^{(k)}\left(\frac{\rho}{r-1+\rho},s\right)/k! \\
F_{m,r+1}(w,s) &= \sum_{k=0}^{n+a+m-r-1}\left(w-\frac{\rho}{r+1+\rho}\right)^k F_{m,r+1}^{(k)}\left(\frac{\rho}{r+1+\rho},s\right)/k!
\end{aligned}
$$

it is then readily seen that

$$F^{(j)}_{m-1,r-1}\left(\frac{\rho}{r+\rho}, s\right) = \sum_{k=j}^{n+a+m-r} \left(\frac{-\rho}{(r+\rho)(r-1+\rho)}\right)^{k-j} F^{(k)}_{m-1,r-1}\left(\frac{\rho}{r-1+\rho}, s\right)/(k-j)!$$

$$F^{(j)}_{m,r-1}\left(\frac{\rho}{r+\rho}, s\right) = \sum_{k=j}^{n+a+m-r-1} \left(\frac{\rho}{(r+\rho)(r+1+\rho)}\right)^{k-j} F^{(k)}_{m,r+1}\left(\frac{\rho}{r+1+\rho}, s\right)/(k-j)!$$

This step by step procedure, lengthy though it may be, in fact [illegible] the full solution to the problem. We shall illustrate it [illegible] the first few steps in a very simple example.

[illegible] steps in a simple example: $n = 1$, $a = 1$, with immigration

[illegible] already known from [1] that

$$F_{01}(w,s) = \left(w - \frac{\rho}{1+\rho}\right)(s+1+\rho)^{-1} + \frac{\rho s^{-1}}{1+\rho}$$

$$F_{00}(w,s) = (s+1+\rho)^{-1}\left\{(w-1)^2(s+2\rho)^{-1} + 2(w-1)(s+\rho)^{-1} + s^{-1}\right\}.$$

[illegible] for $g_0(z,w,t)$, the g.f. associated with $m = 0$, the Laplace transform will be

$$\int_0^\infty e^{-st} g_0(z,w,t)\,dt = \int_0^\infty e^{-(s+\sigma)t} h_0(z,w,t)\,dt = F_{00}(w, s+\sigma) + zF_{01}(w, s+\sigma).$$

We now obtain from (3.13) that

$$F_{10}(w,s) = \sigma\left\{\left(w - \frac{\rho}{2+\rho}\right)(s+2+\rho)^{-1}(s+1+\rho)^{-1} - \frac{\rho s^{-1}(s+1+\rho)^{-1}}{(1+\rho)(2+\rho)} + \frac{\rho s^{-2}}{1+\rho}\right\}$$

$$F_{11}(w,s) = \sigma\left(w - \frac{\rho}{1+\rho}\right)^2(s+1+\rho)^{-1}(s+2(1+\rho))^{-1}\left\{2(s+2+\rho)^{-1} + (s+2\rho)^{-1}\right\} +$$

$$2\sigma\left(w - \frac{\rho}{1+\rho}\right)(s+1+\rho)^{-2}\left\{\frac{2\rho(s+2+\rho)^{-1}}{1+\rho} - \frac{(s+2\rho)^{-1}}{1+\rho} + (s+\rho)^{-1}\right\} +$$

$$\sigma s^{-1}(s+1+\rho)^{-1}\left\{\frac{2\rho^2(s+2+\rho)^{-1}}{(1+\rho)^2} + \frac{(s+2\rho)^{-1}}{(1+\rho)^2} - 2\frac{(s+\rho)^{-1}}{1+\rho} + s^{-1}\right\}$$

$$F_{10}(w,s) = 2\sigma(w-1)^3(s+3\rho)^{-1}(s+1+\rho)^{-1}(s+2(1+\rho))^{-1}\left\{2(s+2+\rho)^{-1}+(s+2\rho)^{-1}\right\}+$$
$$2\sigma(w-1)^2(s+2\rho)^{-1}(s+1+\rho)^{-1}\left\{\frac{(2\rho+1)(s+2(1+\rho))^{-1}}{1+\rho}\left[2(s+2+\rho)^{-1}+(s+2\rho)^{-1}\right]+(s+1+\rho)^{-1}\left[\frac{2\rho(s+2+\rho)^{-1}}{1+\rho}-\frac{(s+2\rho)^{-1}}{1+\rho}+(s+\rho)^{-1}\right]\right\}+$$
$$2\sigma(w-1)(s+\rho)^{-1}(s+1+\rho)^{-1}\left\{\frac{(\rho+3)(s+2(1+\rho))^{-1}}{1+\rho}\left[2(s+2+\rho)^{-1}+(s+2\rho)^{-1}\right]+(s+1+\rho)^{-1}\left[\frac{2\rho(s+2+\rho)^{-1}}{1+\rho}-\frac{(s+2\rho)^{-1}}{1+\rho}+(s+\rho)^{-1}\right]\right\}-$$
$$2\sigma s^{-1}(s+1+\rho)^{-1}\left\{\frac{(s+2(1+\rho))^{-1}}{1+\rho}\left[2(s+2+\rho)^{-1}+(s+2\rho)^{-1}\right]+(s+1+\rho)^{-1}\left[\frac{2\rho(s+2+\rho)^{-1}}{1+\rho}-\frac{(s+\rho)^{-1}}{1+\rho}+(s+\rho)^{-1}\right]\right\}$$

It is clear that the g.f. $g_1(z,w,t)$ associated with $m=1$ will have the Laplace transform

$$\int_0^\infty e^{-st} g_1(z,w,t)\,dt = \int_0^\infty e^{-(s+\sigma)t} h_1(z,w,t)\,dt - F_{10}(w,s+\sigma) + z\,F_{11}(w,s+\sigma) + z^2 F_{12}(w,s+\sigma)\,. \tag{5.4}$$

The procedure outlined may be continued indefinitely for values $m=2,3,\ldots$. We note that while our method is straightforward in principle, it is decidedly tedious in practice, the formulae derived being of such unwieldy lengths as to restrict their usefulness.

The question thus arises as to whether some judicious approximation, such as that suggested by Whittle [4], or some other, may not prove more satisfactory in practice. To this possibility, we turn our attention in a subsequent paper.

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